Modelling Concurrency with Interval Traces

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Motivation

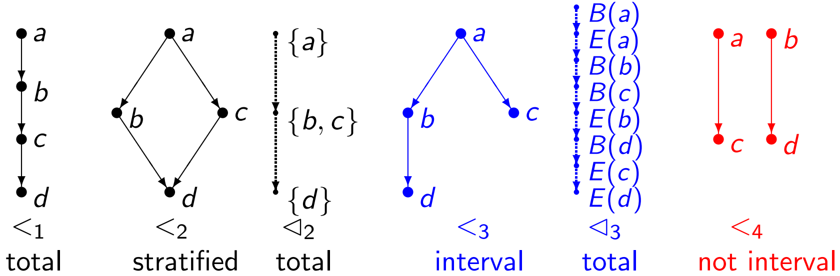
- Most observational semantics of concurrent systems are defined either in terms of sequences (i.e. total orders) or step-sequences (i.e. stratified orders).
- When concurrent histories are fully described by *causality relations*, i.e. *partial orders*, Mazurkiewicz traces [Mazurkiewicz 1977] allow a representation of the entire partial order by a single sequence (plus *independency* relation).
- Other relevant observations can be derived as just stratified or interval extensions of appropriate partial orders.
- However when we want to model both causality and "not later than" relationship, we have to use *stratified order structures* [Gaifman-Pratt 1987, Janicki-Koutny 1991] , when *all* observations are step-sequences, or *interval order structures* [Lamport 1986, Janicki-Koutny 1991], when *all* observations are interval orders.

- *Comtraces* [Janicki-Koutny 1995] allow a representation of stratified order structures by single step-sequences (with appropriate *simultaneity* and *serializability* relations).
- It was argued by Norbert Wiener in 1914 that any execution that can be observed by a single observer must be an interval order.
- It implies that the most precise observational semantics is defined in terms of interval orders.
- However generating interval orders directly is problematic for most models of concurrency. Unfortunately, the only feasible sequence representation of interval order is by using sequences of *beginnings* and *endings* of events involved [Fishburn 1970].

- The goal of this research is to provide a monoid based model that would allow a single sequence of beginnings and endings (enriched with appropriate *independency* relation) to represent the entire *interval order structures* as well as all equivalent interval order observations.

- This will be done by introducing the concept of *interval traces*, and proving that each interval trace uniquely determines an interval order structure.

Interval Orders



- The partial order $<_1$ is an extension of $<_2$, $<_2$ is an extension of $<_3$, and $<_3$ is and extension of $<_4$.
- The total order $<_1$ is uniquely represented by a sequence $abcd$.
- The stratified order $<_2$ is uniquely represented by a step sequence $\{a\}\{b,c\}\{d\}$.
- The interval order $<_3$ is (*not* uniquely) represented by a sequence that represents $\lhd_3$, i.e. $B(a)E(a)B(b)B(c)E(b)B(d)E(c)E(d)$.

Definition

For a relation $R \subseteq X \times X$, any relation $Q \subseteq X \times X$ is an *extension* of R if $R \subseteq Q$.

- $\mathrm{Total}(<) = \{\lhd \subseteq X \times X \mid \lhd$ is a total order and $< \subseteq \lhd\}$.

Theorem (Szpilrajn 1931)

For every partial order $<$,

$$< \;=\; \bigcap_{\lhd \in \mathrm{Total}(<)} \lhd,$$

i.e. each partial order is the intersection of all its total extensions.

Theorem (Fishburn 1970)

A partial order $<$ *on* X *is* interval *iff there exists a total order* $\lhd$ *on some* T *and two mappings* $B, E : X \to T$ *such that for all* $x, y \in X$,

1. $B(x) \lhd E(x)$,
2. $x < y \iff E(x) \lhd B(y)$.

- Usually $B(x)$ is interpreted as the beginning and $E(x)$ as the end of an *interval* x.

Mazurkiewicz Traces

Definition (Mazurkiewicz 1977)

1. Let Σ be a finite set and let the relation $ind \subseteq \Sigma \times \Sigma$ be an irreflexive and symmetric relation (called *independency*). The pair (Σ, ind) is called a *trace alphabet*.
2. Let $\approx \in \Sigma^* \times \Sigma^*$ be a relation defined as follows:
   $x \approx y \iff$
   $\exists x_1, x_1 \in \Sigma^*. \exists (a, b) \in ind.\ x = x_1 abx_2 \wedge y = x_1 bax_2$
3. Let $\equiv_{ind}$ the reflexive and symmetric closure of $\approx$, i.e. $\equiv_{ind} = \approx^*$. Clearly is an *equivalence* relation.
4. For every $x \in \Sigma$, the equivalence class $[x]_{\equiv_{ins}}$ is called a **Mazurkiewicz trace**, or just a **trace**.

Sequences and Partial Orders

Each sequence of events represents a total order of *enumerated events* in a natural way. Let s be a sequence. Then:

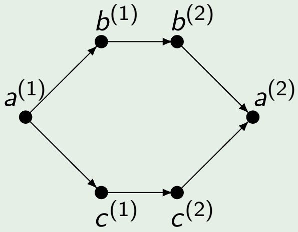
1. $\hat{s}$ denote its enumerated representation. For example if $s = abbaa$ then $\hat{s} = a^{(1)}b^{(1)}b^{(2)}a^{(2)}a^{(3)}$.
2. $\widehat{\Sigma}_s$ denotes the set of all *enumerated events* of s. For example $\widehat{\Sigma}_{abbaa} = \{a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}\}$.
3. For each trace $[s]$, we define $\widehat{\Sigma}_{[s]} = \widehat{\Sigma}_s$.
4. For ever $s \in \Sigma^*$, $\lhd_s$ is a *total order* defined by the enumerated sequence $\widehat{s}$. Fore example
$$\lhd_{abbaa} = a^{(1)} \rightarrow b^{(1)} \rightarrow b^{(2)} \rightarrow a^{(2)} \rightarrow a^{(3)}.$$

Definition

For every trace $[x]$, the partial order
$$\lessdot^{trace}_{[x]} = \bigcap_{s\in[x]} \lhd_s$$
is called the *partial order generated by* $[x]$.

Example

Let $\Sigma = \{a, b, c\}$, $ind = \{(b, c), (c, b)\}$. Given three sequences $s = abcbca$, $s_1 = abc$ and $s_2 = bca$, we can generate the traces $[s] = \{abcbca, abccba, acbbca, acbcba, abbcca, accbba\}$, $[s_1] = \{abc, acb\}$ and $[s_2] = \{bca, cba\}$. Note that $[s] = [s_1][s_2]$ since $[abcbca] = [abc][bca] = [abc\ bca]$. In this case we have *enumerated* events $\widehat{\Sigma}_{[s]} = \{a^{(1)}, b^{(1)}, c^{(1)}, b^{(2)}, c^{(2)}, a^{(2)}\}$, and the partial order $\lessdot^{trace}_{[s]}$ generated by $[s]$ is depicted as Hasse diagram below.



Definition (Lamport 1986, Janicki-Koutny 1991)

An *interval order structure* is a relational structure $S = (X, \prec, \sqsubset)$, such that for all $a, b, c, d \in X$:

I1: $a \not\sqsubset a$

I2: $a \prec b \implies a \sqsubset b$

I3: $a \prec b \prec c \implies a \prec c$

I4: $a \prec b \sqsubset c \;\vee\; a \sqsubset b \prec c \implies a \sqsubset c$

I5: $a \prec b \sqsubset c \prec d \implies a \prec d$

I6: $a \sqsubset b \prec c \sqsubset d \implies a \sqsubset d \;\vee\; a = d$.

The relation $\prec$ is called *causality* and the relation $\sqsubset$ is called *weak causality*.

- In this model the *causality* relation $\prec$ represents the "earlier than" relationship, and the *weak causality* relation $\sqsubset$ represents the "not later than" relationship *but under the assumption that the system runs are interval orders*.

- For every partial order $<$, we define $x <^\frown y \iff \neg(y < x)$, i.e. x is either smaller than y, or they are incomparable, or y is never before x.
- $\prec$ is a partial order,
- if $<$ is an interval order on X, then $(X, <, <^\frown)$ is an interval order structure, i.e. interval orders can be interpreted as simple instances of interval order structures.

Definition

1. An interval order $<$ on X is an *interval extension* of an interval order structure $S = (X, \prec, \sqsubset)$ if $\prec \subseteq <$ and $\sqsubset \subseteq <^\frown$, i.e. if $<$ is an extension of $\prec$ and $<^\frown$ is an extension of $\sqsubset$.
2. The set of all interval extensions of S will be denoted by $\mathrm{interv}(S)$.

Theorem (Janicki-Koutny 1997)

For each interval order structure $S = (X, \prec, \sqsubset)$*, we have*

$$S = \Big(X, \bigcap_{<\in \mathrm{interv}(S)} <, \bigcap_{<\in \mathrm{interv}(S)} <^{\frown}\Big),$$

i.e. S *is entirely defined by the set of all its extensions.*

- The above theorem is a generalization of Szpilrajn's Theorem to interval order structures.
- It is interpreted as proof of the claim that interval order structures uniquely represent sets of equivalent system runs, provided that the system's operational semantics can be fully described in terms of interval orders.

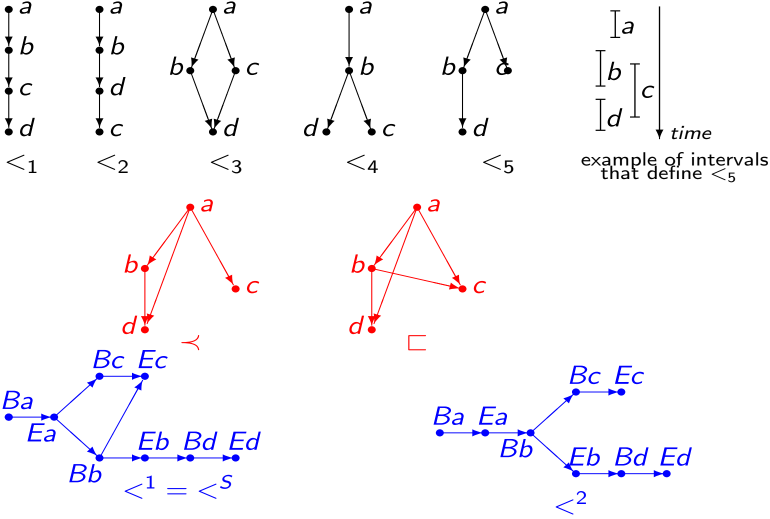
ABM Theorem

Theorem (Abraham, Ben-David, Magidor 1991)

A triple $S = (X, \prec, \sqsubset)$ is an interval order structure if and only if there exists a partial order $<$ on some Y and two mappings $B, E : X \to Y$ such that $B(X) \cap E(X) = \emptyset$ and for each $x, y \in X$:

1. $B(x) < E(x)$,
2. $x \prec y \iff E(x) < B(y)$,
3. $x \sqsubset y \iff B(x) < E(y)$. ∎

- The above theorem, called 'ABM Theorem', can be interpreted as a generalization of Fisburn's Theorem from interval orders to interval order structures.



- An interval order structure $S = (X, \prec, \sqsubset)$, with $X = \{a, b, c, d\}$.
- Interval extensions $\mathrm{interv}(S) = \{<_1, <_2, <_3, <_4, <_5\}$.
- Partial orders $<^1$ and $<^2$ represent the interval order structure S via ABM Theorem. The partial order $<^1$ is $<^S$, the minimal partial order for S that satisfies ABM Theorem.

- Let Σ be a finite set (of events), and let
$$\mathcal{E}_\Sigma = \{Ba \mid a \in \Sigma\} \cup \{Ea \mid a \in \Sigma\},$$
be the set of all beginnings and ends of events in Σ
- For every $\mathcal{D} \subseteq \mathcal{E}_\Sigma$ we define the projection $\pi_\mathcal{D}$ in a standard way, for example: $\pi_{\{Ba,Ea\}}(BbBaEbBaEaEc) = BaBaEa$ and $\pi_{\{Ba,Ea,Bc,Ec\}}(BbBaEbBaEaEc) = BaBaEaEc$.
- We say that a sequence x is *interval* iff
$$\forall Bt, Et \in \mathcal{E}^*.\ \pi_{\{Bt,Et\}}(x) \in (BtEt)^*.$$
- $BbBaEbBaEaEc$ is interval, while $BaBcBbEbEaEc$ is not.

Definition

Let x be an interval sequence, and let $\blacktriangleleft_x$ be a relation on $\widehat{\Sigma}$, defined by
$$a^{(i)} \blacktriangleleft_x b^{(j)} \iff Ea^{(i)} \lhd_x Bb^{(j)}.$$
By Fishburn Theorem, the relation $\blacktriangleleft_x$ is an interval order, and it is called the *interval order defined by the sequence x of beginnings and ends.*

- For example if $x = BaEaBbBcEbBdEcEd$ then $\blacktriangleleft_x$ is the interval order $<_3$ from page 5.

Definition

Let $ind \subseteq \mathcal{E} \times \mathcal{E}$ be a symmetric and irreflexive relation such that for all $a, b \in \Sigma$

1. $(Ba, Ea) \notin ind$ and $(Ea, Ba) \notin ind$,
2. $(Ba, Bb) \in ind$ and $(Ea, Eb) \in ind$.

The relation *ind* is called **interval independency**, and the pair $(\mathcal{E}, ind)$ is called **interval trace alphabet**.

- The interval traces are defined as a special distinctive class Mazurkiewicz traces.

Definition

A trace $[x]_{ind}$ over the interval trace alphabet $(\mathcal{E}, ind)$ is called an **interval trace** if every element of $[x]_{ind}$ is an interval sequence.

- The soundness of the above definition follows from the following non-trivial result.

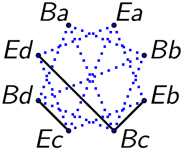
Proposition

Let $(\mathcal{E}, ind)$ be an interval trace alphabet, and let x, y be interval sequences. Then:

1. *xy is an interval sequence.*
2. *Every element of $[x]_{ind}$ is an interval sequence.*
3. *Every element of $[x]_{ind}[y]_{ind} = [xy]_{ind}$ is an interval sequence.*
4. $\blacktriangleleft_x = \blacktriangleleft_y \implies x \equiv_{ind} y$. *(Proof of this is very difficult)*

Example

Consider $\Sigma = \{a, b, c, d\}$, *ind* is the relation described below (the default part of the relation *ind* is represented by blue dotted lines):



and $\mathbf{y}$ is the following set of sequences:

$$\mathbf{y} = \left\{ \begin{array}{l} BaEaBbEbBcEcBdEd, BaEaBbEbBdEdBcEc, BaEaBbBcEbEcBdEd, \\ BaEaBcBbEbEcBdEd, BaEaBcBbEcEbBdEd, BaEaBbBcEcEbBdEd, \\ BaEaBbEbBcBdEcEd, BaEaBbEbBdBcEcEd, BaEaBbEbBdBcEdEc, \\ BaEaBbEbBcBdEdEc, BaEaBbBcEbBdEcEd, BaEaBbBcEbBdEdEc, \\ BaEaBcBbEbBdEdEc, BaEaBcBbEbBdEcEd \end{array} \right\},$$

then $\mathbf{y} = [x]_{ind}$ for any $x \in \mathbf{y}$, for example

$$[x]_{ind} = [BaEaBbEbBcEcBdEd]_{ind}.$$

Definition

For every interval trace $\mathbf{x} = [x]_{ind}$, let $Interv(\mathbf{x}) = \{\blacktriangleleft_t |\ t \in \mathbf{x}\}$ denote the set of all interval orders defined by the elements of $\mathbf{x}$).

For our example, $Interv(\mathbf{y}) = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1, <_2, <_3, <_4$, and $<_5$ are partial orders from from page 15, with $a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}$ represented just by a, b, c, d. In this case

- $BaEaBbEbBcEcBdEd$ represents a total order $<_1$,
- $BaEaBbEbBdEdBcEc$ represents a total order $<_2$,
- each of the sequences $BaEaBbBcEbEcBdEd$, $BaEaBcBbEbEcBdEd$, $BaEaBcBbEcEbBdEd$ and $BaEaBbBcEcEbBdEd$, represents a stratified order $<_3$,
- each of the sequences $BaEaBbEbBcBdEcEd$, $BaEaBbEbBdBcEcEd$, $BaEaBbEbBdBcEdEc$ and $BaEaBbEbBcBdEdEc$ represents a stratified order $<_4$,
- and each of the sequences $BaEaBbBcEbBdEcEd$, $BaEaBbBcEbBdEdEc$, $BaEaBcBbEbBdEdEc$ and $BaEaBcBbEbBdEcEd$ represents the interval $<_5$.

- Assume that a set of events Σ and an interval trace alphabet $(\mathcal{E}, ind)$ are given.
- Recall that for each sequence $x \in \mathcal{E}^*$,
- $\widehat{\mathcal{E}}_x$ is the set of all elements of $\hat{x}$, the enumerated version of x,
- $\lhd_x$ is the *total* order that is equivalent to the sequence x , and
- $\lessdot_{[x]}$ is the partial order that is equivalent to the trace $[x]_{ind}$.

We are now ready to define an interval order structure induced by a single sequence $x \in \mathcal{E}^*$.

Definition

For each $x \in \mathcal{E}^*$, let $S^x = (\widehat{\Sigma}_x^{\mathcal{E}}, \prec_x, \sqsubset_x)$, where $\widehat{\Sigma}_x^{\mathcal{E}} = \{a^{(i)} \mid Ba^{(i)} \in \widehat{\mathcal{E}}_x\} \cup \{a^{(i)} \mid Ea^{(i)} \in \widehat{\mathcal{E}}_x\}$, and $\prec_x$ and $\sqsubset_x$ are relations on $\widehat{\Sigma}_x^{\mathcal{E}}$ defined as follows, for all $a, b \in \Sigma$:

1. $a^{(i)} \prec_x b^{(j)} \iff Ea^{(i)} \lessdot_{[x]} Bb^{(j)}$.
2. $a^{(i)} \sqsubset_x b^{(j)} \iff Ba^{(i)} \lessdot_{[x]} Eb^{(j)}$.

MAIN RESULTS I

Proposition

If x is an interval sequence, then $S^x = (\widehat{\Sigma}_x^{\mathcal{E}}, \prec_x, \sqsubset_x)$ is an interval order structure.

Theorem

For all interval sequences x, y, $x \equiv y$ if and only if $S^x = S^y$.

- The above theorem makes possible the following definition.

Definition

For each interval trace $[x]$, the interval order structure $S^{[x]}$ induced by $[x]$, in defined as $S^{[x]} = (\widehat{\Sigma}_x^{\mathcal{E}}, \prec_{[x]}, \sqsubset_{[x]}) = S^t = (\widehat{\Sigma}_x^{\mathcal{E}}, \prec_t, \sqsubset_t)$, where $t \in [x]$.

MAIN RESULTS II

Theorem

For every interval sequence x,

$$\text{interv}(S^x) = Interv([x]) = \{\blacktriangleleft_t \mid t \in [x]\}.$$

Proposition

For every interval sequence x, $\lessdot_{[x]} \; = <^{S^x}$.

- MOST PROOFS RELY HEAVILY ON ABM THEOREM.
- It can be shown that if all interval orders are stratified, the approach can be reduced to comtraces and stratified order structures as proposed by Janicki and Koutny in 1995.

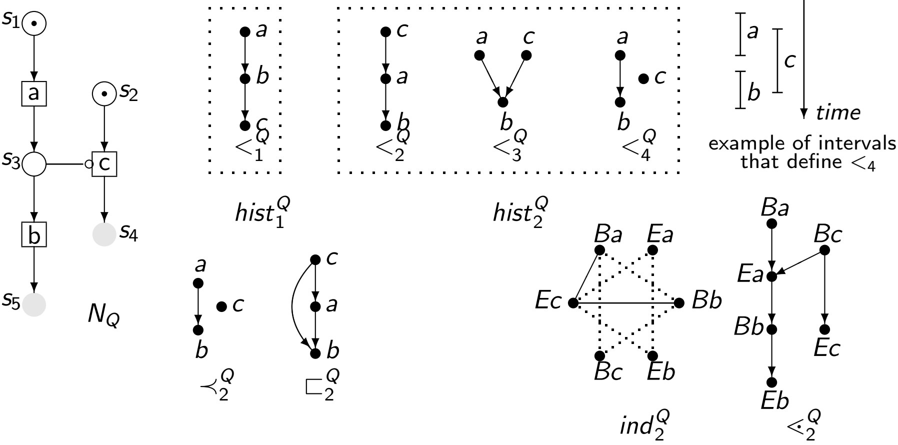
Example

- Let $\Sigma = \{a, b, c, d\}$. Then we have $\mathcal{E} = \{Ba, Ea, Bb, Eb, Bc, Ec, Bd, Ed\}$.
- Let $ind \subseteq \mathcal{E} \times \mathcal{E}$ be the interval independency page 19.
- Take $x = BaEaBbEbBcEcBdEd$. Since x is interval sequence then the interval trace $[x]$ is defined, and $[x] = \mathbf{y}$, where $\mathbf{y}$ is also from page 19 (it contains fourteen sequences).
- The interval order structure $S^{[x]} = S^x$ is $(\widehat{\Sigma}_x^{\mathcal{E}}, \prec, \sqsubset)$, where $\widehat{\Sigma}_x^{\mathcal{E}} = \{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}\}$, and the relations $\prec$ and $\sqsubset$ are these from page 15, after replacing a with $a^{(1)}$, b with $b^{(1)}$, etc.
- The set $\widehat{\mathcal{E}}_x = \{Ba^{(1)}, Ea^{(1)}, Bb^{(1)}, Eb^{(1)}, Bc^{(1)}, Ec^{(1)}, Bd^{(1)}, Ed^{(1)}\}$ and the relation $\lessdot_{[x]} \subseteq \widehat{\mathcal{E}}_x \times \widehat{\mathcal{E}}_x$ equals $<^1$ also from page 15, after replacing Ba with $Ba^{(1)}$, Ea with $Ea^{(1)}$, etc.
- The set $\mathrm{interv}(S^{[x]}) = \{<_1, <_2, <_3, <_4, <_5\}$, where $<_1$, $<_2$, $<_3$, $<_4$ and $<_5$ are interval orders from page 15, again after replacing a with $a^{(1)}$, b with $b^{(1)}$, etc.

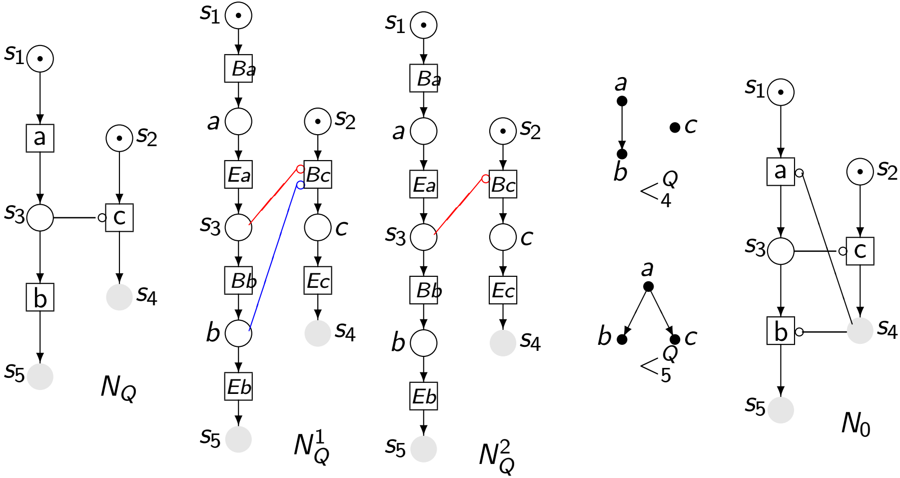
Example - Continued

Moreover

- $<_1 = \blacktriangleleft_{BaEaBbEbBcEcBdEd}$,
- $<_2 = \blacktriangleleft_{BaEaBbEbBdEdBcEc}$,
- $<_3 = \blacktriangleleft_{BaEaBbBcEbEcBdEd} = \blacktriangleleft_{BaEaBcBbEbEcEdEd} = \blacktriangleleft_{BaEaBcBbEcEbBdEd} = \blacktriangleleft_{BaEaBbBcEcEbBdEd}$,
- $<_4 = \blacktriangleleft_{BaEaBbEbBcBdEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEcEd} = \blacktriangleleft_{BaEaBbEbBdBcEdEc} = \blacktriangleleft_{BaEaBbEbBcBdEdEc}$,
- $<_5 = \blacktriangleleft_{BaEaBbBcEbBdEcEd} = \blacktriangleleft_{BaEaBbBcEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEdEc} = \blacktriangleleft_{BaEaBcBbEbBdEcEd}$.
- Finally note that the results would be the same if x would be replaced by any $t \in [x]$.



- N_Q produces two concurrent histories, $hist_1^Q$ and $hist_2^Q$.
- The interval order structure $S_2^Q = (\{a, b, c\}, \prec_2^Q, \sqsubset_2^Q)$ represents the history $hist_2^Q$.
- The independency relation ind^Q can derived the net N_Q.
- The partial order $\lessdot_2^Q$ is generated by the interval trace $[BaBcEaEbEcEb]_{ind^Q}$, where $BaBcEaEbEcEb$ is an interval sequence that represents $<_4^Q$.



- The nets N_Q and N_Q^1 can be regarded as equivalent, but N_Q^2 additionally generates $<_5^Q$, which is not a valid execution of N_Q.
- The net N_0 generates only interval order observations, namely $<_4^Q$.

- Fishburn, P. C.: Intransitive indifference with unequal indifference intervals. Journal of Mathematical Psychology 7, 144–149 (1970).
- H. Gaifman and V. Pratt, Partial Order Models of Concurrency and the Computation of Function, *Proc. of LICS'87* (Logic in Computer Science), pp. 72-85.
- Janicki, R., Koutny, M.: Invariants and Paradigms of Concurrency Theory. In Proc. of PARLE'91. Lecture Notes in Computer Science, vol. 506, pp. 59–74 (1991)
- Janicki, R., Koutny, M.: Structure of Concurrency. Theor. Comput. Sci. 112, 5–52 (1993)
- Janicki, R., Koutny, M.: Semantics of Inhibitor Nets. Inf. Comput. 123(1), 1–16 (1995)

- Janicki, R., Koutny, M.: Fundamentals of Modelling Concurrency Using Discrete Relational Structures. Acta Inform. 34, 367–388 (1997)
- R. Janicki, X. Yin, N. Zubkova, Modeling Interval Order Structures with Partially Commutative Monoids. In Proc. of CONCUR2012, Lecture Notes in Computer Science, vol. 7454, pp. 425-439 (2012)
- Lamport, L.: The mutual exclusion problem: Part I - a theory of interprocess communication; Part II - statements and solutions. Journal of ACM 33(2), 313–326 (1986)
- Mazurkiewicz, A.: Concurrent Program Schemes and Their Interpretation. TR DAIMI PB-78, Comp. Science Depart., Aarhus University (1977)

Wiener, N.: A contribution to the theory of relative position, Proc. of the Cambridge Philosophical Society 17, 441–449 (1914)

Zuberek, W. M.: Timed Petri nets and preliminary performance evaluation. In Proc. of the 7-th Annual Symp. on Computer Architecture, pp. 89–96, La Baule, France (1980)

THANK YOU! QUESTIONS?