A Decomposition Theorem
for
Persistent Labelled Transition Systems

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Labelled transition systems with initial state

\((S, \rightarrow, T, s_0)\) where

- \(S\) are states
- \(T\) are labels
- \(\rightarrow \subseteq (S \times T \times S)\) are the set of arcs
- \(s_0 \in S\) is an initial state.

\[ S = \{s_0, s_1, s_2, s_3\} \]
\[ T = \{a, b, c, d\} \]
\[ \rightarrow = \{(s_0, a, s_1), (s_1, c, s_0), (s_0, b, s_2), (s_2, d, s_0), (s_1, b, s_3), (s_3, d, s_1), (s_2, a, s_3), (s_3, c, s_2)\} \]
Reachability notation

- \(s[t] \) if \( \exists s' \in S: (s, t, s') \in \rightarrow \)
  
  \(t\) is enabled (activated, firable) in state \(s\).

- \(s[t]s'\) if \((s, t, s') \in \rightarrow\).

- Reachability:
  - \(s[\varepsilon]\) and \(s[\varepsilon]s\) are always true
  - \(s[\sigma t]\) iff there is some \(s''\) with \(s[\sigma]s''\) and \(s''[t]s'\)
    \(s[\sigma t]s'\) iff there is some \(s''\) with \(s[\sigma]s''\) and \(s''[t]s'\).

- \([s]\): set of states reachable from \(s\).
Persistency of an lts $(S, \rightarrow, T, s_0)$

Persistency:

\[
s \xrightarrow{t} u \quad \text{implies} \quad \exists r:
\]

Our results are about the cyclic structure of a persistent lts.
Why are persistent systems interesting?

They cover a general notion of conflict-freeness.
Asynchronous Circuits Design People like them.
Every Petri net can be simulated by a persistent net plus two non-persistent transitions.
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Marked graphs are persistent

Note:
All simple cycles have the same Parikh vectors
Parikh vectors

Let \( \sigma = t_1 \ldots t_n \in T^* \) be a finite sequence of labels.

Its Parikh vector is \( \psi(\sigma) : \left\{ \begin{array}{c} T \\ t \end{array} \rightarrow \mathbb{N} \right\} \) number of times \( t \) occurs in \( \sigma \).

Example: \[
\begin{align*}
\psi(t_3 t_1 t_3)(t_1) &= 1 \\
\psi(t_3 t_1 t_3)(t_2) &= 0 \\
\psi(t_3 t_1 t_3)(t_3) &= 2
\end{align*}
\]
Permutations

- Two activated sequences $s[\sigma]$ and $s[\sigma']$ arise from each other by a transposition if

$$\sigma = t_1 \ldots t_k t' t \ldots t_n \quad \text{and} \quad \sigma' = t_1 \ldots t_k t' t \ldots t_n.$$  

- Two activated sequences $s[\sigma]$ and $s[\sigma']$ are permutations of each other (from $s$, written $\sigma \equiv_s \sigma'$) if they arise out of each other through a sequence of transpositions.
Simple cycles

- A cycle from $s$ is a firable sequence that reproduces $s$:
  \[ s[\sigma]s. \]

- A cycle $s[\sigma]s$ is simple if there is no permutation $\sigma \equiv_s \tau_1 \tau_2$ with
  - $\tau_1$ and $\tau_2$ are nontrivial: $\tau_1 \neq \varepsilon \neq \tau_2$
  - $s[\tau_1]s[\tau_2]s.$
Example (non-simple vs. simple cycles)

- $M_0[btbata]M_0$ is not a simple cycle because $btbata \equiv_{M_0} btabta$ and $M_0[bta]M_0[bta]M_0$.
- $M_0[bta]M_0$ is a simple cycle.
A persistent net which is not a marked graph

Note: All simple cycles have the same Parikh vectors or are transition-disjoint
A non-persistent net

Note: There are two simple cycles which do not have the same Parikh vectors and are not transition-disjoint.
Some properties of an lts \((S, \rightarrow, T, s_0)\)

- **finite** if \(S\) and \(T\) are finite
- **deterministic** if \(\forall s \in [s_0], t \in T: s[t]s' \land s[t]s'' \Rightarrow s' = s''\)
- **weakly periodic** if for every \(s_1 \in [s_0], \sigma \in T^*,\) and
  \[
  s_1[\sigma]s_2[\sigma]s_3[\sigma]s_4[\sigma]\ldots,
  \]
  either \(\forall i, j \geq 1: s_i = s_j\) or \(\forall i, j \geq 1: i \neq j \Rightarrow s_i \neq s_j\)
- **cycle-consistent** if for all \(\sigma \in T^*,\)
  \(\exists s \in [s_0]: s[\sigma]s\) implies \(\forall s', s'' \in S: s'[\sigma]s'' \Rightarrow s' = s''\)

Reachability graphs of Petri nets are always deterministic, weakly periodic and cycle-consistent.
Main decomposition theorem

Let \((S, \rightarrow, T, s_0)\) be finite, deterministic, weakly periodic, cycle-consistent, and persistent.

There exists a reachable state \(\tilde{s}\) and a finite set of label-disjoint simple cycles \(\tilde{s}[\rho_i]\tilde{s}\)

such that:

for any reachable state \(s\) and for any cycle \(s[\rho]s\),

\[\Psi(\rho) = \sum k_i \Psi(\rho_i)\]

for some \(k_i \geq 0\).

Note: This statement is wrong in the previous example.
Keller’s theorem

For label sequences $\tau$ and $\sigma$,

\[
\tau \cdot \varepsilon = \tau
\]

\[
\tau \cdot t = \begin{cases} 
\tau, & \text{if there is no label } t \text{ in } \tau \\
\text{the sequence obtained by erasing the leftmost } t \text{ in } \tau, & \text{otherwise}
\end{cases}
\]

\[
\tau \cdot (t\sigma) = (\tau \cdot t) \cdot \sigma.
\]

Keller’s theorem: if an Lts is deterministic and persistent then
Lemma: Let an lts be deterministic and persistent. Let $s[\gamma]$ and $s[\kappa\gamma]$. Then $s[\gamma\kappa']$ with $\Psi(\kappa) = \Psi(\kappa')$ and $\kappa\gamma \equiv_s \gamma\kappa'$. 

Proof: By Keller’s Theorem, $s[\gamma]$ and $s[\kappa\gamma]$ entail $s[\gamma(\kappa\gamma\cdot\gamma)]$. Put $\kappa' = \kappa\gamma\cdot\gamma$. Then $\Psi(\kappa\gamma\cdot\gamma) = \Psi(\kappa)$, hence $\Psi(\kappa\gamma) = \Psi(\gamma\kappa')$. $\kappa\gamma \equiv_s \gamma\kappa'$ follows since both sequences are activated at $s$. 
Existence of home states

**Proposition:** Let an lts be finite, deterministic and persistent. Then $\exists \tilde{s} \in S \forall s \in [s_0]: \tilde{s} \in [s]$.

**Proof:** Let the set of reachable states be $\{s_0, \ldots, s_m\}$. Put $\tilde{s}_0 = s_0$. Select for each $i$ from 1 up to $m$ some state $\tilde{s}_i$ reachable from $\tilde{s}_{i-1}$ and $s_i$, which exists by Keller’s theorem. Then put $\tilde{s} = \tilde{s}_m$. 
Disjointness lemma

**Lemma**: Let an LTS be finite, deterministic, weakly periodic, persistent. Let \( s[\tau]r \) and \( s[\sigma]r \) be two sequences with \( s \neq r \). Then there is at least one label which occurs both in \( \tau \) and in \( \sigma \).

**Proof**: By contraposition, using Keller’s Theorem. If \( \tau \) and \( \sigma \) are label-disjoint, then \( \tau \cdot \sigma = \tau \) and \( \sigma \cdot \tau = \sigma \).

The West and East corners of Diamond 1 and of Diamond 2 are the same by determinacy.

Thus: \( s[\sigma]r[\sigma]q[\sigma] \ldots \)

By weak periodicity and \( s \neq r \), the set of reachable states is infinite.
Lemma 1: about the uniqueness of simple cycles

Lemma 1: Let an LTS be finite, deterministic, weakly periodic, cycle-consistent, and persistent. Let $s[a_\tau]s$ and $s[a_\sigma]s$ with simple $s[a_\tau]s$ and $s[a_\sigma]s$. Then $a_\tau \equiv_s a_\sigma$.
Proof outline of Lemma 1

- Using Keller’s Theorem, $\tau \cdot \sigma = \varepsilon$ implies $\sigma \cdot \tau = \varepsilon$.
- By symmetry, there are two separate cases.
- **Case 1:** $\tau \cdot \sigma = \varepsilon = \sigma \cdot \tau$.
  Then $\Psi(\tau) = \Psi(\sigma)$, implying $\Psi(a\tau) = \Psi(a\sigma)$.
  Then also $a\tau \equiv_s a\sigma$.
- **Case 2:** $\tau \cdot \sigma \neq \varepsilon \neq \sigma \cdot \tau$. Then
  1. The sequences $\sigma \cdot \tau$ and $\tau \cdot \sigma$ are both activated at $s$, and when executed from $s$, they lead to the same state, say to $\tilde{s}$.
  2. $\tilde{s} \neq s$.

By finiteness and the disjointness lemma, $\sigma \cdot \tau$ and $\tau \cdot \sigma$ have some label in common; contradiction.
Reversibility

It would be nice to extend Lemma 1 in the following way:
If two simple cycles $s_1[τ]s_1$ and $s_2[σ]s_2$ have a label in common, then they are Parikh-equivalent.

However, this is true only for reversible lts.

An lts with initial state $s_0$ is called reversible if $∀s ∈ [s_0]: s_0 ∈ [s]$. 
An non-reversible persistent net

Before firing $c$, $M_0[a_1b_1b_2a_2a_3b_3b_4a_4]M_0$ is a simple cycle.

After firing $M_0[c]M$, $M[a_1a_2a_3a_4]M$ and $M[b_1b_2b_3b_4]M$ are simple cycles.
Hypersimple cycles

A cycle $s[\rho]s$ is **hypersimple** if $\Psi(\rho)$ differs from $\Psi(\rho_1) + \Psi(\rho_2)$ for any two non-trivial cycles $s_1[\rho_1]s_1$ and $s_2[\rho_2]s_2$ from reachable markings $s_1$ and $s_2$.

In the previous example,

- $M_0[a1 b1 b2 a2 a3 b3 b4 a4]M_0$ is simple but not hypersimple.
At a home state, every simple cycle is hypersimple

**Lemma**: Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent. Let \( \tilde{s} \in [s_0] \) be a home state. Then every simple cycle \( \tilde{s}[\rho] \tilde{s} \) is hypersimple.

**Proof**: Suppose \( \Psi(\rho) = \Psi(\rho_1) + \Psi(\rho_2) \) for nontrivial cycles \( s_1[\rho_1]s_1 \) and \( s_2[\rho_2]s_2 \) from reachable states \( s_1 \) and \( s_2 \).

Because \( \tilde{s} \) is a home state, \( s_1[\chi]\tilde{s} \) for some label sequence \( \chi \).

By the permutation lemma applied in \( s_1 \) with \( \kappa = \rho_1 \) and \( \gamma = \chi \), \( \Psi(\rho_1) = \Psi(\rho'_1) \) for some cycle \( \tilde{s}[\rho'_1] \tilde{s} \).

By the definition of \( \bullet \), \( \rho'_1 \bullet \rho = \varepsilon \) since \( \Psi(\rho'_1) \leq \Psi(\rho) \).

By Keller’s Theorem, applied to \( \tilde{s}[\rho] \tilde{s} \) and \( \tilde{s}[\rho'_1] \tilde{s} \), \( \tilde{s}[\rho'_1 \bullet \rho] s \) and \( \tilde{s}[\rho \bullet \rho'_1] s \) for some state \( s \), with \( \rho(\rho'_1 \bullet \rho) \equiv \tilde{s} \rho'_1(\rho \bullet \rho'_1) \).

As \( \rho'_1 \bullet \rho = \varepsilon \), \( \tilde{s} = s \) and \( \Psi(\rho) = \Psi(\rho'_1) + \Psi(\rho \bullet \rho'_1) \).

Recalling that \( \Psi(\rho) = \Psi(\rho_1) + \Psi(\rho_2) \), both \( \Psi(\rho'_1) = \Psi(\rho_1) \) and \( \Psi(\rho \bullet \rho'_1) = \Psi(\rho_2) \) differ from the null vector.

Now \( \tilde{s}[\rho'_1] \tilde{s}[\rho \bullet \rho'_1] \tilde{s} \), and therefore \( \tilde{s}[\rho] \tilde{s} \) is not a simple cycle.
Lemma 2: Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent.

Let $s, s'$ be reachable states and $s[\tau\rangle s$ and $s'[\sigma\rangle s'$ be two hypersimple cycles.

If some label $a$ occurs in both cycles, then $\Psi(\tau) = \Psi(\sigma)$. 
Proof of Lemma 2 (part 1 of 2)

By Keller’s Theorem, there exist a state $s''$ and two label sequences $\xi$ and $\chi$ such that $s[\xi]s''$ and $s'[\chi]s''$. By the permutation lemma applied in $s$ with $\gamma = \xi$ and $\kappa = \tau$, there exists a label sequence $\tau'$ such that $s''[\tau']s''$ and $\Psi(\tau) = \Psi(\tau')$. By the permutation lemma applied in $s'$ with $\gamma = \chi$ and $\kappa = \sigma$, there exists a label sequence $\sigma'$ such that $s''[\sigma']s''$ and $\Psi(\sigma) = \Psi(\sigma')$. Let $\tau' = \tau'_1 t \tau'_2$ and $\sigma' = \sigma'_1 t \sigma'_2$ such that $t$ occurs neither in $\tau'_1$ nor in $\sigma'_1$, and let $r$ and $r'$ be the two states such that $s''[\tau'_1]r$ and $s''[\sigma'_1]r'$, respectively. By Keller’s Theorem, applied to $s''[\tau'_1]r$ and $s''[\sigma'_1]r'$, there exists a state $r''$ such that $r[\sigma'_1 \bullet \tau'_1]r''$ and $r'[\tau'_1 \bullet \sigma'_1]r''$. 

Proof of Lemma 2 (part 2 of 2)

By the permutation lemma applied in \( r \) with \( \gamma = \sigma_1' \cdot \tau_1' \) and \( \kappa = t\tau_2'\tau_1' \), there exists a label sequence \( \tau'' \) such that \( r''[\tau'']r'' \) and \( \Psi(\tau'') = \Psi(t\tau_2'\tau_1') = \Psi(\tau) \).

Similarly, there exists a label sequence \( \sigma'' \) such that \( r''[\sigma'']r'' \) and \( \Psi(\sigma'') = \Psi(t\sigma_2'\sigma_1') = \Psi(\sigma) \).

Now \( r[t], r[\sigma_1' \cdot \tau_1']r'' \), and label \( t \) does not occur in \( \sigma_1' \cdot \tau_1' \) since it does not occur in \( \sigma_1' \).

By persistency, \( r''[t]\tilde{r} \) for some state \( \tilde{r} \).

As \( \Psi(t) \leq \Psi(\tau'') = \Psi(\tau), t \cdot \tau'' = \epsilon \).

By Keller’s Theorem, applied to \( r''[\tau'']r'' \) and \( r''[t]\tilde{r}, \tilde{r}[\tau'' \cdot t]r'' \).

As the Parikh vector of \( r''[t]\tilde{r}[\tau'' \cdot t]r'' \) is equal to \( \Psi(\tau'') = \Psi(\tau) \), this cycle is hypersimple.

Similarly, one can construct a hypersimple cycle \( r''[t]\tilde{r}[\sigma'' \cdot t]r'' \).

As every hypersimple cycle is simple and both cycles start with \( t \) from \( r'' \), Lemma 1 applies, entailing \( t(\tau'' \cdot t) \equiv_s t(\sigma'' \cdot t) \) and hence \( \Psi(\tau) = \Psi(\tau'') = \Psi(\sigma'') = \Psi(\sigma) \).
Putting the pieces together

Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent.
There exists a reachable state $\tilde{s}$ and a finite set of label-disjoint simple cycles $\tilde{s}[\rho_i]\tilde{s}$ such that:
for any reachable state $s$ and for any cycle $s[\rho]s$,
$\Psi(\rho) = \sum k_i \Psi(\rho_i)$ for some $k_i \geq 0$.

Roadmap of the proof: Choose some home state $\tilde{s}$. Push $s[\rho]s$ to a Parikh-equivalent cycle $\tilde{s}[\rho']\tilde{s}$.
Permute and decompose $\tilde{s}[\rho']\tilde{s}$ into a sequence of simple cycles through $\tilde{s}$.
Any simple cycle $\tilde{s}[\rho]\tilde{s}$ is hypersimple.
By Lemma 2, two simple cycles through $\tilde{s}$ are either transition-disjoint or Parikh equivalent.
The special case of reversible Petri nets

For reversible, bounded and persistent nets

- the notions of simplicity and hypersimplicity coincide
- and every reachable marking is a home marking.

Decomposition corollary:

Let $N$ be reversible, bounded, and persistent. There is a finite set $B$ of semipositive $T$-invariants such that any two of them are transition-disjoint and every cycle $M[\rho]M$ in the reachability graph decomposes up to permutations to some sequence of cycles $M[\rho_1]M[\rho_2]M \ldots [\rho_n]M$ with all Parikh vectors $\Psi(\rho_i)$ in $B$.

Difference to the decomposition theorem: $M[\rho]M$ can be decomposed already at $M$. 
A consequence of the decomposition corollary

Every bounded, persistent and reversible Petri net $N$ whose unique minimal integral basis $B$ satisfies $|B| = n$ can be viewed (up to reachability graph isomorphism) as the $\oplus$ of $n$ bounded, persistent and reversible Petri nets $N_i$ whose unique minimal integral bases $B_i$ satisfy $|B_i| = 1$.

(Not an immediate corollary.)
The special case of marked graphs

- If there is some nontrivial cycle in the reachability graph of a weakly connected marked graph, then it is automatically reversible. Hence we have a unique basis $B$ as in the decomposition corollary.
- The vector assigning the number 1 to every transition is the only member of $B$. Thus, all Parikh vectors of cycles are multiples of $(1, \ldots, 1)$ (recovering a well-known result).
The interest in this research may lie...

- ...in the proofs of Lemmas 1 and 2, both of which are non-trivial applications of Keller’s fundamental theorem...
- ...in that it describes a rather nice property of the class of transition systems in question, which may have several other consequences that still need to be looked at...
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Separability

Origin and application: Workflow verification [van Hee et al.]

Let $k \in \{1, 2, 3, \ldots \}$ be a number.
Let $M_0$ be an initial marking of a net $N$ such that every place has a multiple of $k$ tokens (0 or $k$ or $2k$ or $\ldots$).
$(N, M_0)$ is called $k$-separable if, for every firable sequence $M_0[\sigma]$, there are $\sigma_1, \ldots, \sigma_k$ such that

$$\forall j, 1 \leq j \leq k: \left( \frac{1}{k} \cdot M_0 \right)[\sigma_j] \text{ and } Parikh(\sigma) = \sum_{j=1}^{k} Parikh(\sigma_j).$$

The vector $Parikh(\sigma)$, for a sequence $\sigma$ of transitions, counts the number of each transition in $\sigma$.

**Theorem:** Marked graphs are separable.
They can thus be viewed as independent copies (direct sums) of $k$ safe marked graphs $\sim \rightarrow$ reduced state space.
$k = 2$, separable
$k = 2$, a separation
$k = 2,$ not separable
$k = 2$, no separation possible
Open question

Are bounded, reversible and persistent Petri nets separable or not?

As a consequence of the consequence of the decomposition corollary, we need only consider the case that there is a single minimal realisable T-invariant.