Parameter Synthesis for Parametric Interval Markov Chains

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MeFoSyLoMa, Evry

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Motivation

- Introduce **parameters** on probabilities:
  - Imprecisions;
  - Robustness;
  - Dimensioning.
- Compute the **whole** set of parameter values ensuring the desired properties.

Outline

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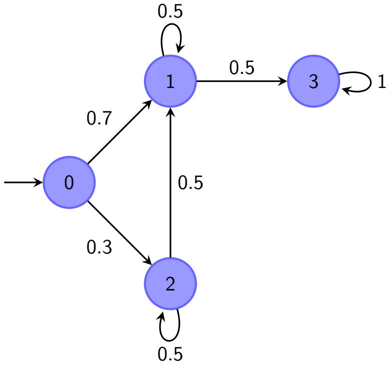
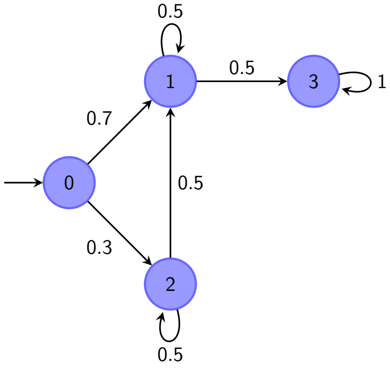
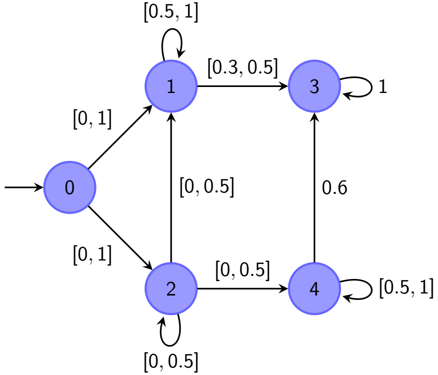
Markov Chains (IMCs)



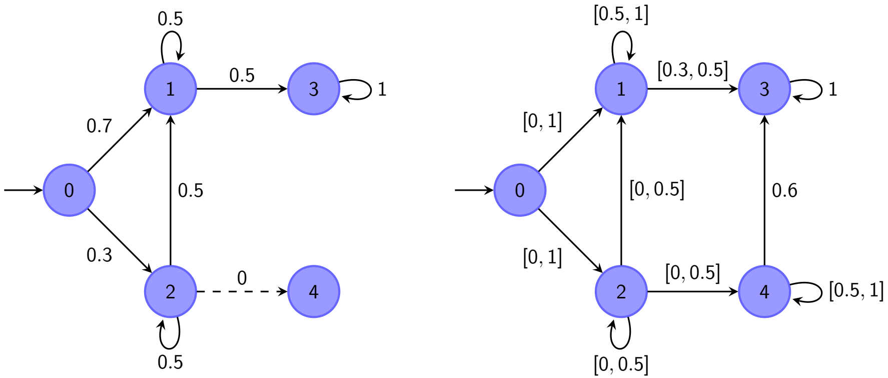
Interval Markov Chains (IMCs)





Specification (IMC)

Markov Chains (IMCs)



Implementation (MC)

With the same structure

Specification (IMC)

Consistency for IMCs

Definition

An IMC is **consistent** if it admits at least one implementation.
Not necessarily with the same structure

Consistency for IMCs

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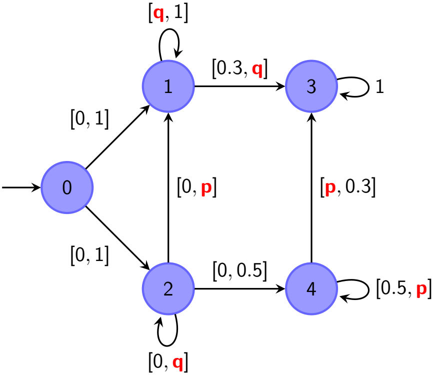
An IMC is **consistent** if it admits at least one implementation.
Not necessarily with the same structure

Theorem ([Delahaye, SynCoP'15])

An IMC is consistent iff it admits an implementation ***with the same structure****.*

So we focus on implementations respecting the structure of the IMC.

Parametric Interval Markov Chains (pIMCs)



Valuating the parameters of $\mathcal{I}$ with valuation v gives an IMC $v(\mathcal{I})$

Consistency for pIMCs

- Does there exists a parameter valuation v such that IMC $v(\mathcal{I})$ is consistent?
- Is IMC $v(\mathcal{I})$ consistent for all parameter valuations v?
- ...

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- ...

Compute all the parameter valuations *v such that IMC $v(\mathcal{I})$ is consistent.*

n-consistency for IMCs

Definition

- State s in an IMC is 0-**consistent** if there exists a probability distribution over the successors of s that matches the intervals;
- State s in an IMC is n-**consistent** ($n \geq 1$) if:
  1. there exists a probability distribution ρ over the successors of s that matches the intervals and
  2. the successors s' such that $\rho(s') > 0$ are $(n-1)$-consistent.

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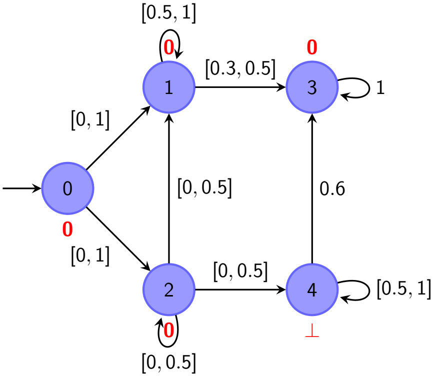
Theorem

An IMC with N states is consistent iff its initial state is N-consistent.

n-consistency for IMCs



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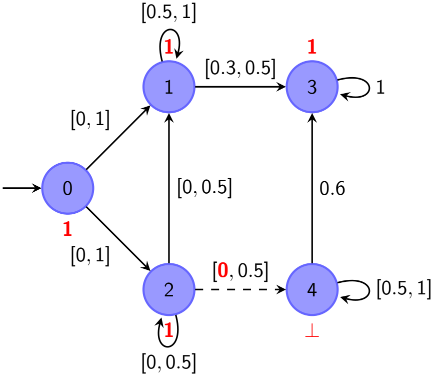

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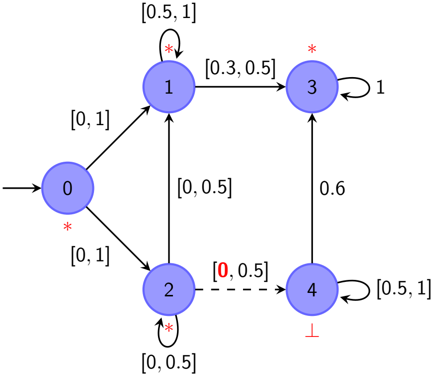

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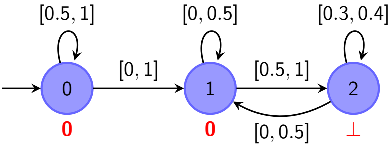

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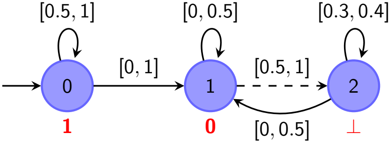

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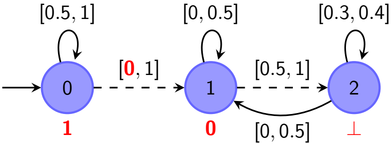

n-consistency constraints for pIMCs

- **Local consistency constraint for state** s wrt. some subset S' of its successors:

$$LC(s,S') = \left[\sum_{s' \in S'} \mathtt{Up}(s,s') \geq 1\right] \cap \left[\sum_{s' \in S'} \mathtt{Low}(s,s') \leq 1\right] \cap \left[\bigcap_{s' \in S'} \mathtt{Low}(s,s') \leq \mathtt{Up}(s,s')\right]$$

n-consistency constraints for pIMCs

- ***n*-consistency for *s* constraint given some cut-off successors**:
  $\texttt{Cons}_0^X(s) = LC(s, \texttt{Succ}(s) \setminus X) \cap \left[\bigcap_{s' \in X} \texttt{Low}(s, s') = 0\right]$

  and for $n \geq 1$,

$$\texttt{Cons}_n^X(s) = \left[\bigcap_{s' \in \texttt{Succ}(s) \setminus X} \texttt{Cons}_{n-1}(s')\right] \cap [LC(s, \texttt{Succ}(s) \setminus X)] \cap \left[\bigcap_{s' \in X} \texttt{Low}(s, s') = 0\right]$$

- ***n*-consistency constraint for *s***:

$$\texttt{Cons}_n(s) = \bigcup_{X \subseteq Z(s)} \texttt{Cons}_n^X(s)$$

$Z(s)$ contains the successors of s for which Low is either 0 or a parameter

Back to Consistency for pIMCs

Theorem

Given a pIMC $\mathcal{I}$ with N states and initial state s_0, and a parameter valuation v:

$$v(\mathcal{I}) \text{ is consistent iff } v \in \mathit{Cons}_N(s_0)$$

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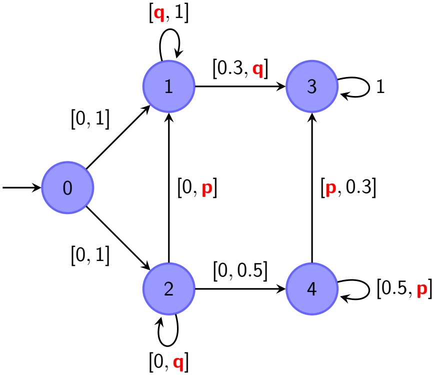

$$[(q \leq 0.7) \cap (q \geq 0.3)] \cup (q = 1)$$

Back to Consistency for pIMCs

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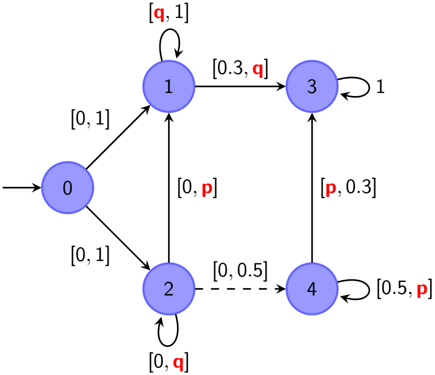

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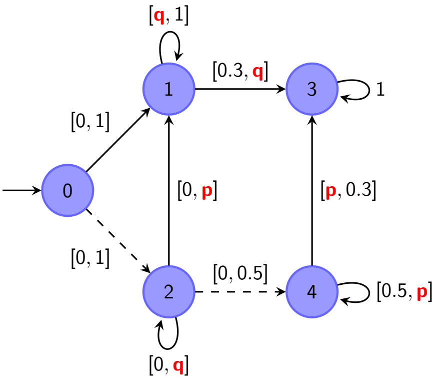

$$[(\mathbf{q} \leq \mathbf{0.7}) \cap (\mathbf{q} \geq \mathbf{0.3})] \cup (q = 1)$$

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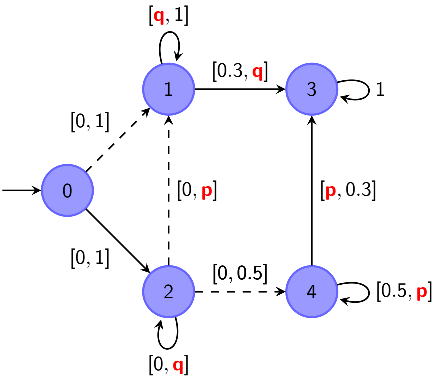

$$[(q \leq 0.7) \cap (q \geq 0.3)] \cup (\mathbf{q} = \mathbf{1})$$

Outline

(Existential) Consistent Reachability for IMCs and pIMCs

Given goal subset of states G:

- **for an IMC**:

  Does the IMC admit an implementation that reaches G (with positive probability);

- **for a pIMCs** $\mathcal{I}$:

  Compute all the parameter valuations *v such that $v(\mathcal{I})$ satisfies (existential) consistent reachability for G.*

(Existential) Consistent Reachability for IMCs and pIMCs

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Theorem

Every implementation of an IMC $\mathcal{I}$ that reaches (states corresponding to) G can be turned into an implementation **respecting the structure** *of $\mathcal{I}$ that reaches G.*

We again focus on implementations respecting the structure of the (p)IMC

n-reachability for IMCs

Definition

- Goal G is **0-reachable** from state s if s is consistent and $s \in G$
- Goal G is **n-reachable** from state s (for $n \geq 1$) if there exists a probability distribution ρ over the successors of s such that:
  1. ρ matches the intervals;
  2. $\rho(s') > 0$ implies s' is consistent;
  3. either $s \in G$ or there exists a successor s'' such that $\rho(s'') > 0$ and G is $(n-1)$-reachable from s''.

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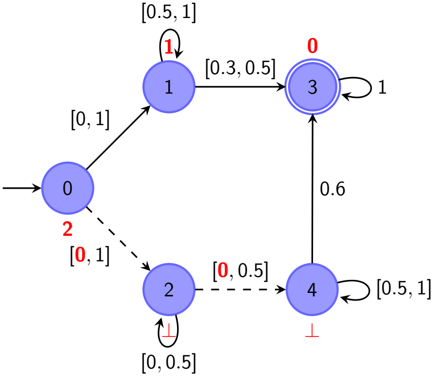
Lemma

Given an IMC $\mathcal{I}$ with initial state s_0 and N states. Goal G is existential consistent reachable in IMC $\mathcal{I}$ iff G is N-reachable from s_0.

n-reachability for IMCs



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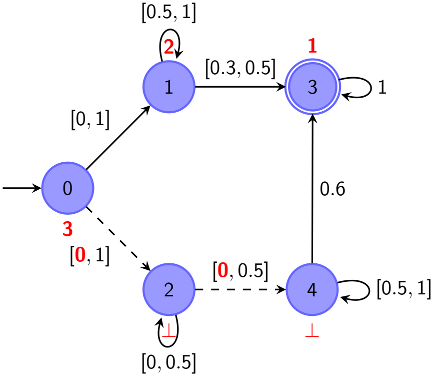

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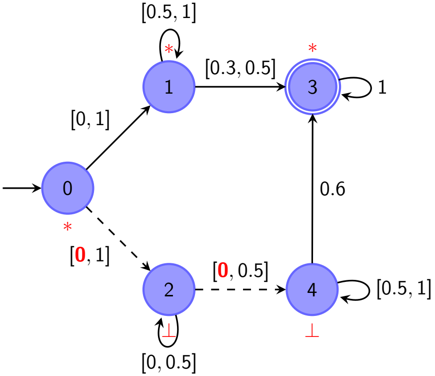

n-reachability for pIMCs

- **Constraint for n-reachability from state s, given a set of cut-off successors**:
  $\texttt{Reach}_0^{G,X}(s) = \texttt{Cons}^X(s) \cap (s \in G)$,

  and for $n > 0$

$$\texttt{Reach}_n^{G,X}(s) = \texttt{Cons}^X(s) \cap$$
$$\left[(s \in G) \cup \bigcup_{s' \in \texttt{Succ}(s) \setminus X} \texttt{Reach}_{n-1}^{G}(s') \cap \texttt{Up}(s,s') > 0 \cap \sum_{s'' \neq s'} \texttt{Low}(s,s'') < 1 \right]$$

- **Constraint for n-reachability from state s**:

$$\texttt{Reach}_n^{G}(s) = \bigcup_{X \subseteq Z(s)} \texttt{Reach}_n^{G,X}(s)$$

Back to Existential Consistent Reachability for pIMCs

Theorem

Given a pIMC $\mathcal{I}$ with N states and initial state s_0, goal subset of states G, and a parameter valuation v:

$$G \text{ is existential consistent reachable in } v(\mathcal{I}) \text{ iff } v \in \mathtt{Reach}_N(s_0)$$

Back to Existential Consistent Reachability for pIMCs

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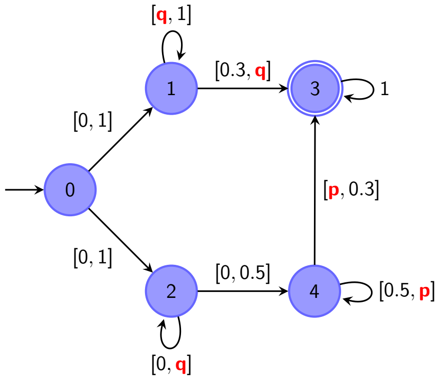

$$[(q \leq 0.7) \cap (q \geq 0.3)]$$

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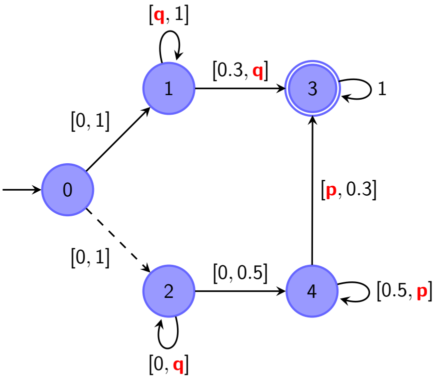

$$[(q \leq 0.7) \cap (q \geq 0.3)]$$

Outline

Parametric IMCs and Consistency

Consistent Reachability

Consistent Avoidability and Universal Consistent Reachability

Conclusion

Consistent Avoidability for IMCs

Given a subset of states G to avoid:

- **for an IMC**:

  Does the IMC admit an implementation that never reaches G (with probability 1);

- **for a pIMC** $\mathcal{I}$:

  Compute all the parameter valuations *v such that $v(\mathcal{I})$ satisfies consistent avoidability for G.*

Consistent Avoidability for IMCs

Given a subset of states G to avoid:

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  Does the IMC admit an implementation that never reaches G (with probability 1);

- **for a pIMC** $\mathcal{I}$:

  Compute all the parameter valuations *v such that $v(\mathcal{I})$ satisfies consistent avoidability for G.*

This is **consistency** where being in G also implies being inconsistent:
replace $LC(s, S')$ by $LC(s, S') \cap (s \notin G)$ in $\texttt{Cons}(s)$.
Call this $\texttt{Avoid}^G(s)$

Universal Consistent Reachability for IMCs

Given a subset of states G to reach:

- **for an IMC**:

  Do all implementations of the IMC reach G (with positive probability)?

- **for a pIMC** $\mathcal{I}$:

  Compute all the parameter valuations *v such that $v(\mathcal{I})$ satisfies universal consistent reachability for G.*

Universal Consistent Reachability for IMCs

Given a subset of states G to reach:

- **for an IMC**:

  Do all implementations of the IMC reach G (with positive probability)?

- **for a pIMC $\mathcal{I}$**:

  Compute all the parameter valuations *v such that $v(\mathcal{I})$ satisfies universal consistent reachability for G.*

$$\mathtt{uReach}^G(s) = \mathtt{Cons}(s) \cap \overline{\mathtt{Avoid}^G(s)}$$

Outline

Parametric IMCs and Consistency

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Conclusion

Conclusion and Future Work

- **Summary**:
  - Algorithms to **effectively compute** the **exact** sets of parameter valuations ensuring consistency in a pIMCs, with reachability-related constraints;
  - The algorithms have exponential complexity.
- **Future work**:
  - More efficient (distributed) algorithms, heuristics;
  - Constraint programming based algorithms;
  - Quantified probabilities;
  - More complex properties.