The Complexity of Concurrent Rational Synthesis

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The Synthesis problem

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Out plays action in B

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Out wins if $\rho \models \varphi$

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Theorem

The synthesis problem for LTL is 2 EXPTIME *complete.*

The Rational Synthesis

$A_0 \uplus A_1 \uplus \ldots \uplus A_n$ sets of actions, φ_0 LTL formula.

P_0 :

P_1 :

$\vdots$

P_n :

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ρ is good if $\rho \models \varphi_0$ or $\rho \not\models \varphi_0$ and $(\sigma_0, \ldots, \sigma_n)$ is not a *rational behavior*

Nash Equilibria

Definition

- A profile σ is a vector of strategies.

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Nash equilibrium is a profile σ such that:

$$\forall i \in \mathsf{N},\ \forall \tau \in \Sigma_i\ , \mathsf{Payoff}_i((\tau, \sigma_{-i})) \leq \mathsf{Payoff}_i(\sigma)\ .$$

- N is the set of players.
- Σ_i is the set of possible strategies for player i.

0-fixed Nash Equilibria

Definition

$0 - NE$ is a profile (σ_0, σ_{-0}) such that:

$$\forall i \in \mathsf{N} \setminus \{0\},\ \forall \tau \in \Sigma_i\ ,$$
$$\mathsf{Payoff}_i((\sigma_0, \sigma_{-0})) \geq \mathsf{Payoff}_i((\sigma_0, \sigma_{-(0,i)}, \tau))\ .$$

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The Rational Synthesis

$A_0 \uplus A_1 \uplus \ldots \uplus A_n$ sets of AP, $\varphi_0, \varphi_1, \ldots, \varphi_n$ LTL formulas.

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Theorem (KPV 14)

The rational synthesis problem for LTL is $2\,\mathsf{EXPTIME}$ *complete.*

ω-Regular Objectives

Natural question

What is the complexity if the objective is Büchi ?

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The original proof

The following SL formula:

$$\psi \equiv \bigwedge_{i=1}^{k} [\![\tau_i]\!] \left(\flat(\sigma_{-i}, \tau_i)(\varphi_i \rightarrow \flat(\sigma)\varphi_i) \right)$$

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Alternative approach

Provide a direct game theoretic proof.

Formal setting

The game graph

Let $\mathcal{G} = (\mathsf{St}, s_0, \mathsf{N}, (\mathsf{Act}_i)_{i \in \mathsf{N}}, \mathsf{Tab})$ be a concurrent game structure, Where :

- St is the set of states in the game,
- s_0 is the initial state,
- $\mathsf{N} = \{0, 1, ..., n\}$ is the set of players,
- Act_i is the set of actions of Player,
- $\mathsf{Tab} : \mathsf{St} \times \prod_{i \in \mathsf{N}} \mathsf{Act}_i \to \mathsf{St}$ is the transition table.

Strategies

A *strategy* for Player i is a mapping :

$$\sigma_i : \mathsf{St}\left(\prod_{i \in \mathsf{N}} \mathsf{Act}_i \ \mathsf{St}\right)^* \to \mathsf{Act}_i \quad .$$

Strategies and payoffs

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Payoffs

Given a profile σ,

$$\mathsf{Payoff}_i(\sigma) = 1 \text{ if } \rho \models \varphi_i$$

where ρ is the play induced by σ.

NCRS Problem

Is there a strategy σ_0 for player 0 such that :

$$\forall \sigma_{-0}, (\sigma_0, \sigma_{-0}) \in 0\text{–}NE, \mathsf{Payoff}_0(\sigma_0, \sigma_{-0}) = 1 \ \ ?$$

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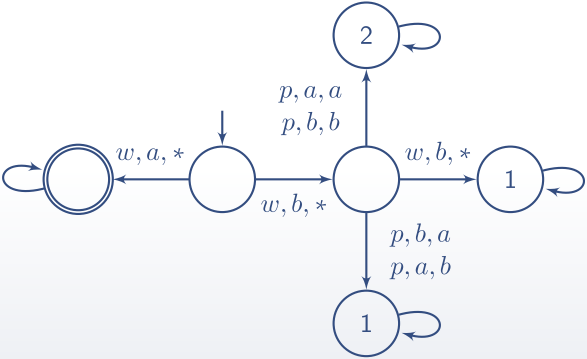

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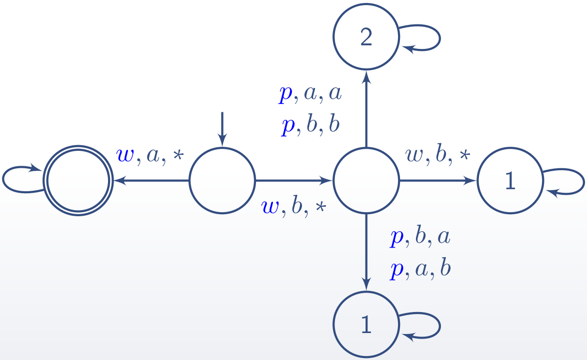

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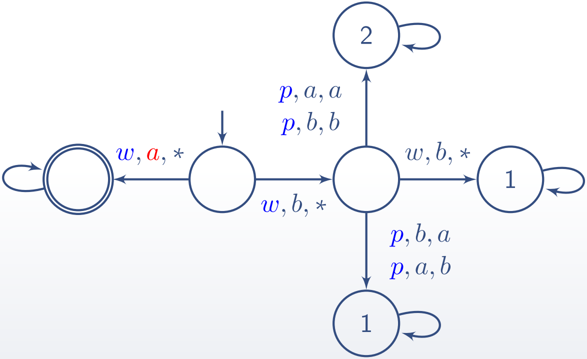

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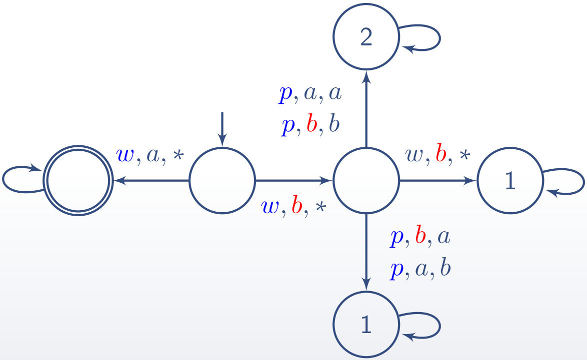

General Approach

Zero-Sum game

We design a game between two players: *Constructor* and *Spoiler* such that:

- Constructor designs a solution σ_0 for Player 0,
- Spoiler tries to spoil σ_0.
  - devices a play ρ that is consistent with σ_0 s.t. $\rho \not\models \phi_0$ and $\rho \in 0 - NE$.
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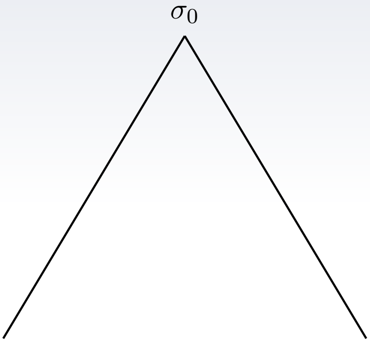
Key idea

In each state of the zero-sum game, we keep track of :

- The current state,
- The set of players that can ensure payoff 1,
- The set of players that can unilaterally deviate.

The solution



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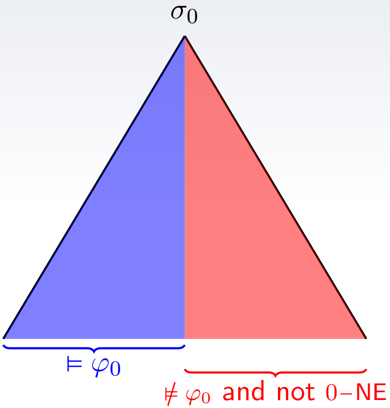

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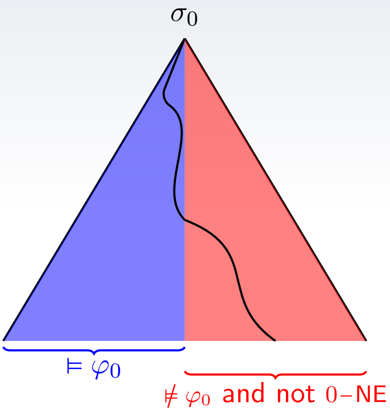

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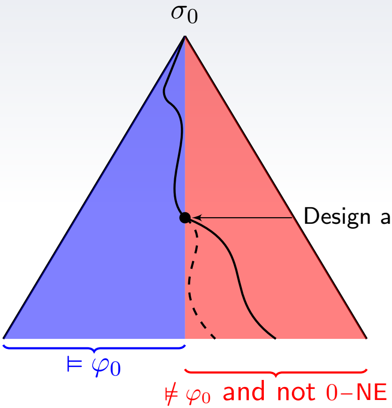

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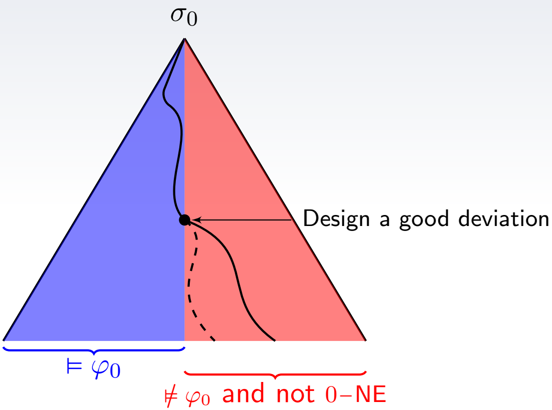

Construction of a good deviation

- Choose a player i s.t. $\text{Payoff}_i(\rho) = 0$,
- Pick actions for 0 that allows i to deviate,
- Pick winning actions for i.

The zero-sum game

In each state we define two sets of players

- D : Players that could deviate unilaterally,
- W : Players with payoff 1.

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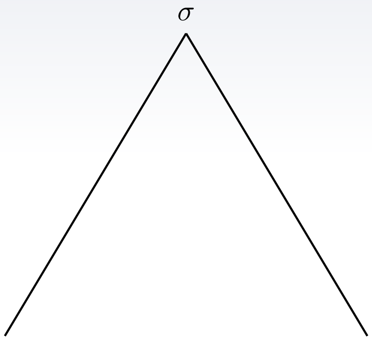
- D : Players that could deviate unilaterally,
- W : Players with payoff 1.

A play ρ is winning if

$$\rho \vDash \varphi_0 \text{ or } \exists i \in D \text{ s.t. } \mathsf{Payoff}_i(\rho) = 0$$
$$\text{and}$$
$$\forall j \in W, \mathsf{Payoff}_j(\rho) = 1 \quad .$$

Correctness

Let σ be a winning strategy for Controller.



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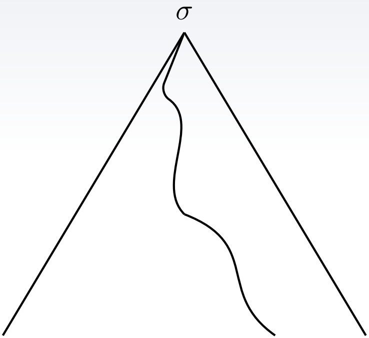

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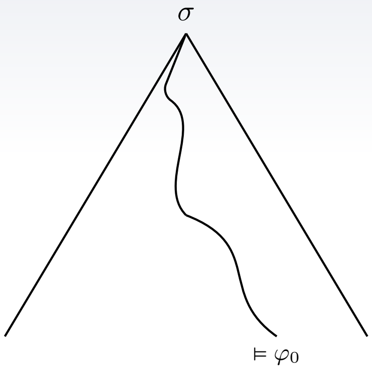

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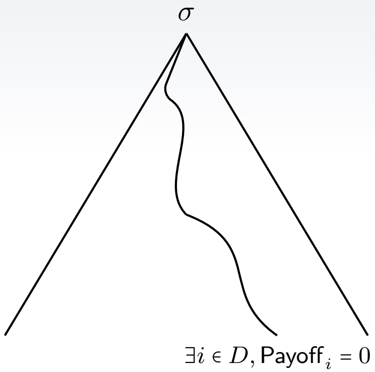

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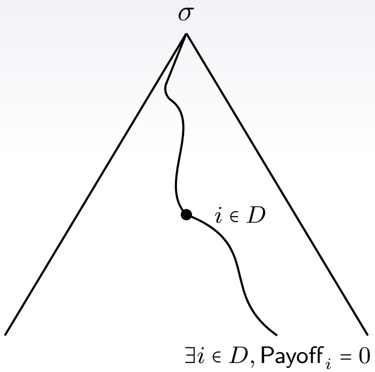

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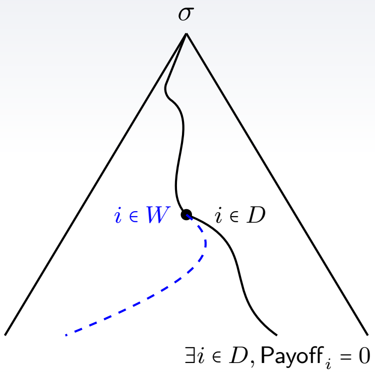

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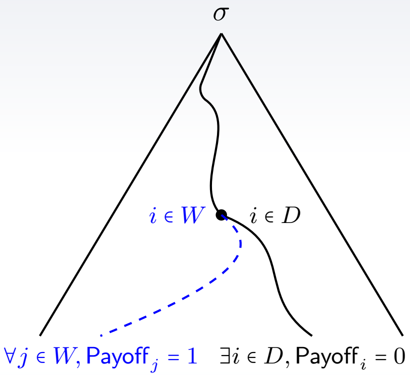

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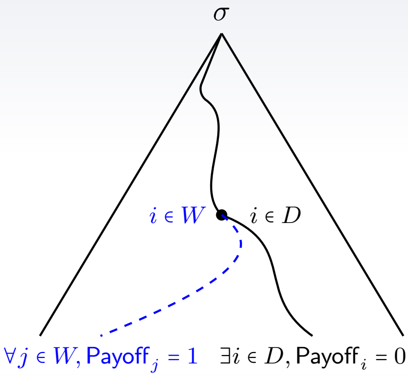

Remark

- Controller chooses actions for every $i \in W$.

The Negotiation Game

1. Constructor assings actions for Player 0 and every player in W.
2. Spoiler proposes a profile for every player in $\mathsf{N} \setminus \{0\}$.
3. Constructor can guess a deviation for a player not in $W \cup D$.
4. Spoiler has the opportunity to change the profile he proposed.

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- If they agree over the choices of steps 3 and 4, then we update W,
- If not we update D.

Main Results

Theorem

Constructor wins iff there exists a solution for the NCRS problem.

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Complexity result

	Perfect Information Concurrent Rational Synthesis Problem	
	Cooperative	Non-Cooperative
Safety	NP [BBMU15] NP-h [CFGR16]	PSpace, PSpace-h [CFGR16]
Reachability	NP [BBMU15] NP-h [CFGR16]	PSpace, PSpace-h [CFGR16]
Büchi	PTime-c [BBMU15]	PSpace, PSpace-h [CFGR16]
co-Büchi	NP [BBMU15] NP-h [CFGR16]	PSpace, PSpace-h [CFGR16]
Parity	$P_{\|\|}^{NP}$ [BBMU15], NP-h [Umm08]	ExpTime, PSpace-h [CFGR16]
Streett	PSpace [BBMU15], NP-h[Umm08]	ExpTime,PSpace-h [CFGR16]
Rabin	$P_{\|\|}^{NP}$ [BBMU15], NP-h [CFGR16], coNP-h [CFGR16]	ExpTime, PSpace-h [CFGR16]
Muller	PSpace [BBMU15], PSpace-h [CFGR16]	ExpTime, PSpace-h [CFGR16]
LTL	2ExpTime-c [FKL10]	2ExpTime-c [KPV14]

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