The Complexity of Concurrent Rational Synthesis

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The Synthesis problem

\[ A \cup B \] two sets, \( \varphi \) LTL formula.
The Synthesis problem

\[ A \cup B \text{ two sets, } \varphi \text{ LTL formula.} \]

Out plays action in \( B \)

In plays action in \( A \)
The Synthesis problem

\[ A \uplus B \text{ two sets, } \varphi \text{ LTL formula.} \]

Out plays action in \( B \)

\[ a_0 \]

In plays action in \( A \)
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Out plays action in \( B \)

\[ \rho : a_0 \ b_0 \ a_1 \ b_1 \ \cdots \]

In plays action in \( A \)

Out wins if \( \rho \models \varphi \)
The Synthesis problem

$A \cup B$ two sets, $\varphi$ LTL formula.

Out plays action in $B$

$\rho : a_0 \; b_0 \; a_1 \; b_1 \; \ldots$

In plays action in $A$

Out wins if $\rho \models \varphi$

Theorem

The synthesis problem for LTL is $2\text{EXPTIME}$ complete.
The Rational Synthesis

\[ A_0 \uplus A_1 \uplus \ldots \uplus A_n \] sets of actions, \( \varphi_0 \) LTL formula.

\begin{align*}
P_0 : \\
P_1 : \\
\vdots \\
P_n : 
\end{align*}
$A_0 \uplus A_1 \uplus \ldots \uplus A_n$ sets of actions, $\varphi_0$ LTL formula.

\[
P_0 : \quad a_0^0
\]

\[
P_1 : \quad a_0^1
\]

\[
\vdots
\]

\[
P_n : \quad a_0^n
\]
The Rational Synthesis

\( A_0 \uplus A_1 \uplus \ldots \uplus A_n \) sets of actions, \( \varphi_0 \) LTL formula.

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P_0 : \quad a_0^0 \quad a_1^0
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\vdots
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Induces a play \( \rho \)
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\end{align*}

Induces a play $\rho$

$\rho$ is good if $\rho \models \varphi_0$ or $\rho \not\models \varphi_0$ and $(\sigma_0, \ldots, \sigma_n)$ is not a rational behavior.
Nash Equilibria

Definition

- A profile $\sigma$ is a vector of strategies.
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Definition

Nash equilibrium is a profile $\sigma$ such that:

$$\forall i \in N, \ \forall \tau \in \Sigma_i, \ Payoff_i((\tau, \sigma_{-i})) \leq Payoff_i(\sigma).$$

- $N$ is the set of players.
- $\Sigma_i$ is the set of possible strategies for player $i$. 
0-fixed Nash Equilibria

Definition

0 – NE is a profile \((\sigma_0, \sigma_{-0})\) such that:

\[
\forall i \in N \setminus \{0\}, \; \forall \tau \in \Sigma_i ,
\text{Payoff}_i((\sigma_0, \sigma_{-0})) \geq \text{Payoff}_i((\sigma_0, \sigma_{-(0,i)}), \tau).
\]

- \(N\) is the set of players.
- \(\Sigma_i\) is the set of possible strategies for player \(i\).
The Rational Synthesis

$A_0 \cup A_1 \cup \ldots \cup A_n$ sets of AP, $\varphi_0, \varphi_1, \ldots, \varphi_n$ LTL formulas.

$$
\sigma_0 \quad P_0 : \quad a_0^0 \quad a_1^0 \quad \ldots \\
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Theorem (KPV 14)

The rational synthesis problem for LTL is $2 \text{EXPTIME complete}$. 
ω-Regular Objectives

Natural question

What is the complexity if the objective is Büchi?

The original proof

The following SL formula:

\[ \psi \equiv \bigwedge_{i=1}^{\infty} \text{disj} \left( \tau_i \right) \left( \mathbf{♭} \left( \sigma - i, \tau_i \right) \left( \phi_i \rightarrow \mathbf{♭} \left( \sigma \right) \phi_i \right) \right) \]

is in a decidable fragment.

Alternative approach

Provide a direct game theoretic proof.
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Alternative approach

Provide a direct game theoretic proof.
Formal setting

The game graph

Let $\mathcal{G} = (\text{St}, s_0, N, (\text{Act}_i)_{i \in N}, \text{Tab})$ be a concurrent game structure, Where:

- $\text{St}$ is the set of states in the game,
- $s_0$ is the initial state,
- $N = \{0, 1, \ldots, n\}$ is the set of players,
- $\text{Act}_i$ is the set of actions of Player $i$,
- $\text{Tab} : \text{St} \times \prod_{i \in N} \text{Act}_i \rightarrow \text{St}$ is the transition table.
Strategies and payoffs

Strategies

A strategy for Player $i$ is a mapping:

$$
\sigma_i : \operatorname{St}\left( \prod_{i \in N} \operatorname{Act}_i \ \operatorname{St} \right)^* \to \operatorname{Act}_i.
$$

Payoffs

Given a profile $\sigma$, payoff $i(\sigma) = 1$ if $\rho/\text{uni22A7} \phi_i$ where $\rho$ is the play induced by $\sigma$.
Strategies and payoffs

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Payoffs

Given a profile $\sigma$,

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\text{Payoff}_i (\sigma) = 1 \text{ if } \rho \models \varphi_i
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where $\rho$ is the play induced by $\sigma$. 
Non Cooperative Rational Setting

NCRS Problem

Is there a strategy $\sigma_0$ for player 0 such that:

$$\forall \sigma_-, (\sigma_0, \sigma_-) \in 0-NE, \text{Payoff}_0(\sigma_0, \sigma_-) = 1$$
Non Cooperative Rational Setting

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Example
Non Cooperative Rational Setting

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Example
General Approach

Zero-Sum game

We design a game between two players: Constructor and Spoiler such that:

- Constructor designs a solution $\sigma_0$ for Player 0,
- Spoiler tries to spoil $\sigma_0$.
  - devices a play $\rho$ that is consistent with $\sigma_0$ s.t. $\rho \neq \phi_0$ and $\rho \in 0 - NE$.
- Constructor replies by designing a profitable deviation from $\rho$ for some player $i$. 
General Approach

Zero-Sum game

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  - devices a play $\rho$ that is consistent with $\sigma_0$ s.t.
    $\rho \neq \phi_0$ and $\rho \in 0 - NE$.
  
- Constructor replies by designing a profitable deviation from $\rho$ for some player $i$.

Key idea

In each state of the zero-sum game, we keep track of:

- The current state,
- The set of players that can ensure payoff 1,
- The set of players that can unilaterally deviate.
The solution

\( \sigma_0 \)
The solution

\[ \sigma_0 \]

Design a good deviation \( \phi_0 \) and not \( 0 - NE \)

Construction of a good deviation

Choose a player \( i \) s.t. Payoff \( i(\rho) = 0 \), Pick actions for \( 0 \) that allows \( i \) to deviate, Pick winning actions for \( i \).

\[ \models \varphi_0 \]

\( \not\models \varphi_0 \) and not \( 0 - NE \)
The solution

\[ \sigma_0 \]

Design a good deviation \( \phi_0 \) and not \( 0 \). 

Construction of a good deviation

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The solution

\[ \sigma_0 \]

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\[ \neq \varphi_0 \] and not 0-NE

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The solution

Construction of a good deviation

- Choose a player $i$ s.t. Payoff$_i(\rho) = 0$,
- Pick actions for 0 that allows $i$ to deviate,
- Pick winning actions for $i$. 
The zero-sum game

In each state we define two sets of players

- $D$: Players that could deviate unilaterally,
- $W$: Players with payoff 1.
The zero-sum game

In each state we define two sets of players
- \(D\) : Players that could deviate unilaterally,
- \(W\) : Players with payoff 1.

A play \(\rho\) is winning if

\[
\rho \models \varphi_0 \text{ or } \exists i \in D \text{ s.t. } \text{Payoff}_i(\rho) = 0
\]

and

\[
\forall j \in W, \text{Payoff}_j(\rho) = 1.
\]
Correctness

Let $\sigma$ be a winning strategy for Controller.
Correctness

Let $\sigma$ be a winning strategy for Controller.

$\sigma$

Remark
Controller chooses actions for every $i \in W$. 

$\exists i \in D, \text{Payoff}_i = 0$

$\forall j \in W, \text{Payoff}_j = 1$
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Let $\sigma$ be a winning strategy for Controller.

$\forall j \in W, \text{Payoff}_j = 1$

$\exists i \in D, \text{Payoff}_i = 0$

Remark: Controller chooses actions for every $i \in W$. 

$\exists i \in D, \text{Payoff}_i = 0$
Correctness

Let $\sigma$ be a winning strategy for Controller.

$\forall j \in W, \text{Payoff}_j = 1 \quad \exists i \in D, \text{Payoff}_i = 0$
Correctness

Let $\sigma$ be a winning strategy for Controller.

$\forall j \in W$, Payoff$_j = 1$

$\exists i \in D$, Payoff$_i = 0$

Remark

- Controller chooses actions for every $i \in W$. 
The Negotiation Game

1. Constructor assigns actions for Player 0 and every player in $W$.
2. Spoiler proposes a profile for every player in $N \setminus \{0\}$.
3. Constructor can guess a deviation for a player not in $W \cup D$.
4. Spoiler has the opportunity to change the profile he proposed.
The Negotiation Game

1. Constructor assigns actions for Player 0 and every player in $W$.
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3. Constructor can guess a deviation for a player not in $W \cup D$.
4. Spoiler has the opportunity to change the profile he proposed.

- If they agree over the choices of steps 3 and 4, then we update $W$.
- If not we update $D$. 
Main Results

Theorem

Constructor wins iff there exists a solution for the NCRS problem.
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**Theorem**

Constructor wins iff there exists a solution for the NCRS problem.

**Complexity result**

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